ISRAEL JOURNAL OF MATHEMATICS 115 (2000), 101–123

BI-GALOIS OBJECTS OVER THE TAFT ALGEBRAS

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ABSTRACT

Let k be a field. We study the groupoid of Hopf bi-Galois objects, whose objects are k-Hopf algebras, and whose morphisms are L-H-bi-Galois extensions of k for Hopf algebras L and H.

We show that if $H = H_{N,m}$ is one of the Taft algebras, then $L \cong H_{N,m}$ in any L-H-bi-Galois object. We compute the group of bi-Galois objects over the two-generator Taft algebras $H_{N,1}$. We classify the isomorphism classes of Galois extensions of k over the general Taft algebras $H_{N,m}$, and we compute the groups of bi-Galois objects over $H_{N,m}$ for odd N.

Our computations for the two-generator Taft algebras rely on Masuoka's classification [9] of their cleft extensions. To treat the general Taft algebras, we will generalize a result of Kreimer [6] to give a description of the Galois objects over a tensor product of two Hopf algebras.

1. Introduction

Hopf-Galois extensions were introduced by Chase and Sweedler [1] (for the commutative case) and Kreimer and Takeuchi [8] as a generalization of the Galois theory of rings [2], in which the action of the Galois group is replaced by the coaction of a Hopf algebra. By definition, an H-Galois extension A of a subalgebra B over a Hopf algebra H is an extension of algebras in which A is a right H-comodule algebra via the coaction $\rho\colon A \to A \otimes H$, the algebra B is the algebra of coinvariants $A^{\operatorname{co} H} := \{a \in A \mid \rho(a) = a \otimes 1\}$, and a certain canonical map $\kappa_A\colon A \otimes_B A \to A \otimes H$ is a bijection. In the present paper, only H-Galois

Received November 19, 1997 and in revised form November 4, 1998

extensions of the base field k are considered; we will call them H-Galois objects for short.

If H is a finite dimensional cocommutative Hopf algebra, then the isomorphism classes of H-Galois objects form a group by [5]. Composition in this group is the cotensor product over H, and if H is the group algebra of a commutative group, then the Harrison group is a subgroup. The construction of a cotensor product Galois object from two H-Galois objects fails if H is not cocommutative. The situation can be amended by considering additional structures. By definition an L-H-bi-Galois object A, for two Hopf algebras H and L, is a right H-Galois object, as defined above, as well as a left L-Galois object (involving a left coaction $A \to L \otimes A$) in such a way that the two coactions make A an L-H-bicomodule. Now, given three Hopf algebras L, H, and R, an L-H-bi-Galois object A, and an H-R-bi-Galois object B, one can form the cotensor product $A \Box_H B$, which is an L-R-bi-Galois object. In this way, one obtains the groupoid of bi-Galois objects, whose objects are Hopf algebras, and whose morphisms are isomorphism classes of bi-Galois objects, with the cotensor product as composition [12]. As a special case, the isomorphism classes of H-H-bi-Galois objects form a group $\mathrm{BiGal}(H)$; this special case was discussed in [17].

A bi-Galois structure is in fact no additional requirement of a Galois object: Whenever A is an H-Galois object, there is a unique Hopf algebra $L = L(A, H)$ associated to it such that A is in fact an L-H-bi-Galois object (see [12], generalizing the commutative case treated by Van Oystaeyen and Zhang [17]). The construction $L(A, H)$ generalizes the double twist of a Hopf algebra by a two-cocycle, as introduced by Doi [3]. The motivation in [17] is that the additional structure of a bi-Galois extension that comes automatically with every Galois object admits the formulation of analogues of the Fundamental Theorem of Galois Theory for Hopf-Galois objects (see also [12, 14] for more general versions).

In the present paper we compute explicitly for an interesting class of Hopf algebras the group $\mathrm{BiGal}(H)$ of bi-Galois objects. The Taft algebras were introduced [16] as an early nontrivial example of a noncommutative non-cocommutative Hopf algebra. The Taft algebra $H_{N,m}$ is generated by m grouplike and one skew primitive element, using a primitive N-th root of unity in the commutation relations between grouplikes and the skew primitive. The two-generator Taft algebras $H_N = H_{N,1}$ occur as building blocks in the finite quotients of quantized enveloping algebras at a root of unity.

Masuoka [9] has classified all cleft extensions over the two-generator Taft algebras. Using his results we compute, in Section 2, all bi-Galois objects over the two-generator Taft algebras. We find that in any L-H-bi-Galois object with $H = H_N$, we also have $L \cong H_N$. Equivalently, every double twist of H_N by a two-cocycle is isomorphic to H_N. We compute the group $\mathrm{BiGal}(H_N)$ to be isomorphic to the semidirect product $\dot{k} \ltimes k$, where $\dot{k}$ denotes the multiplicative group of the base field.

In Section 3 we generalize partly a theorem of Kreimer [6, Thm. 3.7] that describes the group $\mathrm{Gal}(H_1 \otimes H_2)$ of Galois objects over the tensor product of two finitely generated projective cocommutative Hopf algebras H_1 and H_2 (in Kreimer's paper, as well as in Section 3, k is a commutative ring rather than, in the rest of the paper, a field). Taking away the statement on the group structure of $\mathrm{Gal}(H_1 \otimes H_2)$, Kreimer's result says that every Galois object over $H_1 \otimes H_2$ can be built up uniquely (up to isomorphism, of course) from Galois objects A_i over H_i for $i = 1, 2$ and a Hopf algebra pairing $H_1 \otimes H_2 \to k$. We can carry this result over to the non-cocommutative case by taking into account the structure of a bi-Galois object that every Galois object carries naturally by [12]: If H_1 and H_2 are two (not necessarily cocommutative) Hopf algebras, every Galois object over $H_1 \otimes H_2$ can be built up from Galois objects A_i over H_i for $i = 1, 2$, and a skew pairing $L(A_2, H_2) \otimes L(A_1, H_1) \to k$. One can also compute the bi-Galois structure of the Galois object obtained in this way.

The results of Section 3 are put to use when we compute the Galois and bi-Galois objects over the general Taft algebras in the last section. In fact it is easy to see that the Taft algebra $H_{N,m}$ is isomorphic to the tensor product Hopf algebra $kC_N^{m-1} \otimes H_N$, where C_N denotes the cyclic group of order N. Again, we find that in every L-H-bi-Galois object with $H = H_{N,m}$ we have $L \cong H_{N,m}$ as well; equivalently, every cocycle double twist of $H_{N,m}$ is isomorphic to $H_{N,m}$. We compute the group $\mathrm{BiGal}(H_{N,m})$ of bi-Galois objects over the general Taft algebras to be isomorphic to

$$\left(\mathrm{GL}_{m-1}(\mathbb{Z}/(N)) \ltimes (\mathbb{Z}/(N))^{m-1}\right) \ltimes_\omega \left(\mathrm{Skew}_{m-1}(\mathbb{Z}/(N)) \times (\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^{m-1}\right)$$

for suitable actions and a suitable group cocycle ω. Here we denote by $\mathrm{Skew}_{m-1}(\mathbb{Z}/(N))$ the additive group of skew symmetric $(m-1)$-by-$(m-1)$ matrices with zero diagonal.

Preliminaries and Notations. We collect some conventions as well as background material on Hopf-Galois and bi-Galois extensions.

Throughout the paper (except in Section 3) k denotes a fixed base field, and $\dot{k}$ denotes its unit group; algebras, tensor products, coalgebras etc. are over k. We use Sweedler's notation for comultiplication in the form $\Delta(h) = h_{(1)} \otimes h_{(2)}$. We use the notations $\rho(v) = v_{(0)} \otimes v_{(1)}$ for the comodule structure $\rho\colon V \to V \otimes H$ of a right H-comodule. We use $\lambda(v) = v_{(-1)} \otimes v_{(0)}$ for a left H-comodule structure $\lambda\colon V \to H \otimes V$. For a right (resp. left) H-comodule V over a bialgebra H we denote by

$$V^{\mathrm{co}\,H} := \{v \in V \mid \rho(v) = v \otimes 1\} \quad (\text{resp. } {}^{\mathrm{co}\,H}V := \{v \in V \mid \lambda(v) = 1 \otimes v\})$$

the subspace of right (resp. left) coinvariant elements. A (right or left) H-comodule algebra for a bialgebra H is a (right or left) H-comodule and algebra A, so that the comodule structure is an algebra map. The right H-comodule algebra A is said to be an H-Galois extension of its subalgebra B if $B = A^{\mathrm{co}\,H}$ and the map $\kappa_A\colon A \otimes_B A \to A \otimes H$ given by $\kappa_A(x \otimes y) = xy_{(0)} \otimes y_{(1)}$ is a bijection. An H-Galois extension of the base field k will be called an H-Galois object for short. A left H-Galois object is a left H-comodule algebra A such that ${}^{\mathrm{co}\,H}A = k$ and $\kappa'\colon A \otimes A \to H \otimes A$ with $\kappa'(x \otimes y) = x_{(-1)} \otimes x_{(0)}y$ is a bijection. An L-H-bi-Galois object for Hopf algebras L and H is a left L-Galois object as well as right H-Galois object A such that the left and right comodule structures make A an L-H-bicomodule. If A is a right H-Galois object, then there is a unique Hopf algebra $L = L(A, H)$ such that A is an L-H-bi-Galois object. If A is an L-H-bi-Galois object and B is an H-R-bi-Galois object for Hopf algebras L, H, and R, then the cotensor product

$$A \square_H B = \left\{ \sum x_i \otimes y_i \in A \otimes B \,\middle|\, \sum \rho(x_i) \otimes y_i = \sum x_i \otimes \lambda(y_i) \right\},$$

as an L-subcomodule, R-subcomodule, and subalgebra of $A \otimes B$, is an L-R-bi-Galois object. Bi-Galois objects form a groupoid with Hopf algebras as objects, isomorphism classes of bi-Galois objects as morphisms, and cotensor product as composition. In particular, the isomorphism classes of H-H-bi-Galois objects form a group $\mathrm{BiGal}(H)$ for every Hopf algebra H.

A right H-comodule algebra A is said to be **cleft** if there is a convolution invertible right colinear map $\Phi\colon H \to A$. A cleft right H-comodule algebra with $A^{\mathrm{co}\,H} = k$ is a right H-Galois object; the converse is true if H is finite dimensional [7].

For a Hopf algebra H, we denote by $\mathrm{Aut}_{\mathrm{Hopf}}(H)$ the group of its Hopf algebra automorphisms. For an algebra map $u \in \mathrm{Alg}(H, k)$ we denote by $\mathrm{coinn}(u)\colon H \to H$ the coinner automorphism of H induced by u, that is,

$$\mathrm{coinn}(u)(h) = u^{-1}(h_{(1)})h_{(2)}u(h_{(3)}),$$

where $u^{-1} = uS$ denotes the convolution inverse of u. The set $\mathrm{CoInn}(H) := \{\mathrm{coinn}(u) \mid u \in \mathrm{Alg}(H, k)\}$ of coinner automorphisms of H is a normal subgroup of $\mathrm{Aut}_{\mathrm{Hopf}}(H)$; we call the quotient $\mathrm{CoOut}(H) := \mathrm{Aut}_{\mathrm{Hopf}}(H)/\mathrm{CoInn}(H)$ the co-outer automorphism group of H. For a left H-comodule V and a coalgebra map $f\colon H \to H$ we denote by ${}^fV$ the vector space V with the left H-comodule structure $v \mapsto f(v_{(-1)}) \otimes v_{(0)}$, where $v \mapsto v_{(-1)} \otimes v_{(0)}$ denotes the original H-comodule structure of V. A group homomorphism $\mathrm{Aut}_{\mathrm{Hopf}}(H) \to \mathrm{BiGal}(H)$ is given by $f \mapsto {}^fH$. By [12, Lem. 3.11] this induces an injective homomorphism $I\colon \mathrm{CoOut}(H) \to \mathrm{BiGal}(H)$.

Let H be a Hopf algebra. A two-cocycle on H is a map $\sigma\colon H\otimes H \to k$ satisfying

$$\sigma(f_{(1)}, g_{(1)})\sigma(f_{(2)}g_{(2)}, h) = \sigma(g_{(1)}, h_{(1)})\sigma(f, g_{(2)}h_{(2)})$$

and $\sigma(h,1) = \sigma(1,h) = 1$ for all $f,g,h \in H$. If σ is a (convolution) invertible two-cocycle, then Doi [3] constructs a double twisted Hopf algebra H^σ which is H as a coalgebra with multiplication $g \cdot h = \sigma(g_{(1)}, h_{(1)})g_{(2)}h_{(2)}\sigma^{-1}(g_{(3)}, h_{(3)})$.

The right H-comodule algebra $k_\sigma[H] := k\#_\sigma H$ is defined to be H with multiplication defined by $g\cdot h = \sigma(g_{(1)}, h_{(1)})g_{(2)}h_{(2)}$. It is an H^σ-H-bicomodule algebra via the comultiplication of H. If σ is a convolution invertible two-cocycle, then $k_\sigma[H]$ is an H-cleft extension of k. One has $L(k_\sigma[H], H) \cong H^\sigma$.

2. Bi-Galois objects over the two-generator Taft algebras

Let $N > 1$ be an integer, and assume that k contains a primitive N-th root of unity q. We denote by $\langle q\rangle$ the multiplicative group of N-th roots of unity in k, which is generated by q. The two-generator Taft algebra [16] is

$$H_N = H_{N,q} := k\langle X, Y\rangle/(X^N - 1, Y^N, YX - qXY).$$

It is a Hopf algebra with grouplike X and $(1,X)$-primitive Y; that is, comultiplication is given by $\Delta(X) = X \otimes X$ and $\Delta(Y) = 1 \otimes Y + Y \otimes X$, the counit is given by $\varepsilon(X) = 1$ and $\varepsilon(Y) = 0$ and the antipode is given by $S(X) = X^{N-1}$ and $S(Y) = -q^{-1}X^{N-1}Y$. A k-basis of H_N is $\{X^iY^j \mid 0 \le i < N,\ 0 \le j < N\}$. The set of grouplike elements of H_N is $\mathcal{G}(H_N) = \{1, X, X^2, \ldots, X^{N-1}\}$. For any i,j with $j \not\equiv i+1 \bmod N$, all (X^i, X^j)-primitives are trivial (multiples of $X^i - X^j$), while the set of (X^i, X^j)-primitives is $P_{X^i,X^j} = k(X^i - X^j) \oplus kX^iY$ for $j \equiv i + 1 \bmod N$. (See [11, Expl. 1.4] and [10, Lem. 16.1].)

LEMMA 2.1: *We have a group isomorphism*

$$\varphi_0\colon \dot{k} \ni r \mapsto f_r \in \mathrm{Aut}_{\mathrm{Hopf}}(H_N)$$

where $f_r(X) = X$ *and* $f_r(Y) = rY$.

Moreover, $\mathrm{Alg}(H_N, k) = \{u_r | \ r \in \langle q\rangle\}$, *where* $u_r(X) = r$ *and* $u_r(Y) = 0$. *For* $r \in \langle q\rangle$ *we have* $\varphi_0(\mathrm{coinn}(u_r)) = f_r$, *so that* φ_0 *induces an isomorphism*

$$\varphi\colon \dot{k}/\langle q\rangle \to \mathrm{CoOut}(H_N).$$

Proof: Since X is the unique grouplike admitting a nontrivial $(1, X)$-primitive, it is fixed by any Hopf algebra automorphism f. Moreover, $f(Y)$ is a nontrivial $(1, X)$-primitive, hence $f(Y) = rY + t(X - 1)$ for some $r \in \dot{k}$ and $t \in k$, and $0 = f(YX - qXY) = f(Y)X - qXf(Y) = t(1 - q)(X - 1)X$, whence $t = 0$. The remaining assertions are straightforward to check. ∎

The following is a special case of [9, Prop. 2.17, Lem. 2.19]:

THEOREM 2.2: 1. *For* $\alpha \in \dot{k}$ *and* $\beta \in k$ *the algebra*

$$A_{\alpha,\beta} := A_{N,\alpha,\beta} := k\langle x, y\rangle/(x^N - \alpha, y^N - \beta, yx - qxy)$$

is a cleft right H_N*-Galois object with right comodule structure* $\rho = \rho_{\alpha,\beta}$ *defined by* $\rho(x) = x \otimes X$ *and* $\rho(y) = 1 \otimes Y + y \otimes X$. *A right colinear convolution invertible map* $\Phi\colon H_N \to A_{N,\alpha,\beta}$ *is given by* $\Phi(X^iY^j) = x^iy^j$ *for* $0 \leq i < N$ *and* $0 \leq j < N$.

2. *Any right* H_N*-Galois object is of the form described in 1.*
3. *The set* $\mathrm{Alg}^{H_N}(A_{N,\alpha',\beta'}, A_{N,\alpha,\beta})$ *of comodule algebra homomorphisms (all of which are necessarily isomorphisms) is empty if* $\beta \neq \beta'$. *If* $\beta = \beta'$, *it consists precisely of the* g_s *for* $s \in \dot{k}$ *with* $\alpha' = s^N\alpha$, *where* $g_s(x) = sx$ *and* $g_s(y) = y$.

DEFINITION AND LEMMA 2.3: *Let* $\alpha \in \dot{k}$ *and* $\beta \in k$. *Then* $A_{N,\alpha,\beta}$ *is an* H_N*-bi-Galois object with the right comodule structure defined above and the left comodule structure* $\lambda = \lambda_{\alpha,\beta}$ *defined by* $\lambda(x) = X \otimes x$ *and* $\lambda(y) = 1 \otimes y + Y \otimes x$.

Proof: That λ is well defined is proved in the same way as that ρ is well defined (or the comultiplication of H_N is): For any elements a, b of a k-algebra R we have

$$ba = qab \implies (a + b)^N = a^N + b^N$$

which shows that $(1 \otimes y + Y \otimes x)^N = \beta$, while $(X \otimes x)^N = \alpha$ is obvious as well as $(1 \otimes y + Y \otimes x)(X \otimes x) = q(X \otimes x)(1 \otimes y + Y \otimes x)$. That $A := A_{N,\alpha,\beta}$ is an H_N-bicomodule is easy to check. To see that it is left Galois, it is enough to show that the left Galois map $\kappa' = (H_N \otimes \nabla)(\lambda \otimes A)\colon A \otimes A \to H_N \otimes A$ is surjective. For this in turn it is enough to check that for each h in a generating set of H_N there is $\sum x_i \otimes y_i \in A \otimes A$ with $\kappa'(\sum x_i \otimes y_i) = h \otimes 1$. Indeed, if we have $\sum x_i \otimes y_i, \sum x_i' \otimes y_i' \in A \otimes A$ with $\kappa'(\sum x_i \otimes y_i) = h \otimes 1$ and $\kappa'(\sum x_i' \otimes y_i') = g \otimes 1$, then $\kappa'(\sum x_i x_j' \otimes y_j' y_i) = hg \otimes 1$, and for $a \in A$ we have $h \otimes a = \kappa'(\sum x_i \otimes y_i a)$. Now $X \otimes 1 = \kappa'(x \otimes x^{-1})$ and $Y \otimes 1 = \kappa'(y \otimes x^{-1} - 1 \otimes yx^{-1})$ completes the proof. ∎

By [12, Thm. 3.5] there is, for any right H-Galois object A, a unique up to isomorphism Hopf algebra $L(A, H)$ such that A is an $L(A, H)$-H-bi-Galois object.

COROLLARY 2.4: *For any right H_N-Galois object A we have $L(A_{N,\alpha,\beta}, H_N) \cong H_N$. For any invertible two-cocycle σ: $H_N \otimes H_N \to k$ we have $H_N^\sigma \cong H_N$. The Hopf algebra H_N is determined up to isomorphism by the k-linear monoidal category of its comodules.*

The first assertion is contained in the preceding lemma. For an invertible cocycle σ we have $H_N^\sigma \cong L(k\#_\sigma H_N, H_N)$ by [12, Thm. 3.9]. By Theorem 2.2 we have $k\#_\sigma H_N \cong A_{N,\alpha,\beta}$ for suitable α, β, and by Lemma 2.3 we have $L(A_{N,\alpha,\beta}, H_N) \cong H_N$. If H' is a Hopf algebra whose category of comodules is equivalent as a monoidal category to the category of H_N-comodules, then there is an H'-H_N-bi-Galois object by [12, Cor. 5.7.], whence $H' \cong H_N$.

THEOREM 2.5: *The map*

$$\psi\colon \dot{k} \ltimes k \longrightarrow \mathrm{BiGal}(H_N)$$
$$(\alpha, \beta) \longmapsto [A_{N,\alpha,\beta}]$$

is a group isomorphism. The diagram

$$\begin{array}{ccc} \mathrm{CoOut}(H_N) & \xrightarrow{\ I\ } & \mathrm{BiGal}(H_N) \\ \Big\uparrow{\scriptstyle\varphi} & & \Big\uparrow{\scriptstyle\psi} \\ \dot{k}/\langle q\rangle & \xrightarrow[\ I'\ ]{} & \dot{k} \ltimes k \end{array}$$

commutes for $I'([r]) = (r^n, 0)$.

Proof: Let A be any H_N-bi-Galois object. By Theorem 2.2 we can assume that $A = A_{N,\alpha,\beta}$ as a right H_N-comodule algebra, with a suitable left comodule structure λ'. By [12, Thm. 3.5] there is a unique automorphism f: $H_N \to H_N$ with $(f \otimes 1)\lambda = \lambda'$, that is, $A = {}^f A_{N,\alpha,\beta}$ as bicomodule algebras. Let $r \in \dot{k}$ with

$f = f_r$. We claim that the right colinear isomorphism g_r: $A_{N,r^n\alpha,\beta} \to {}^{f_r}A_{N,\alpha,\beta}$ is also left colinear. In fact

$$\begin{aligned}\lambda' g_r(y) &= (f_r \otimes 1)\lambda(y) = (f_r \otimes 1)(1 \otimes y + Y \otimes x) = 1 \otimes y + rY \otimes x \\ &= 1 \otimes y + Y \otimes rx = (1 \otimes g_r)\lambda(y)\end{aligned}$$

and $\lambda' g_r(x) = rX \otimes x = (1 \otimes g_r)\lambda(x)$. We have shown that ψ is onto, and that the diagram in the theorem commutes.

Assume that g: $A_{N,\alpha',\beta'} \to A_{N,\alpha,\beta}$ is an isomorphism of bi-Galois objects. By Theorem 2.2 we have $\beta' = \beta$, $g = g_r$ for some $r \in \dot{k}$ and $\alpha' = r^n\alpha$. Left colinearity of g implies

$$1 \otimes y + Y \otimes x = \lambda g_r(y) = (1 \otimes g_r)\lambda(y) = 1 \otimes y + Y \otimes rx$$

whence $r = 1$ and $\alpha' = \alpha$, showing injectivity of ψ.

To show that ψ is a group homomorphism, we need a homomorphism of H_N-bicomodule algebras

$$\delta\colon A_{N,\alpha\alpha',\beta\alpha'+\beta'} \to A_{N,\alpha,\beta}\square_{H_N} A_{N,\alpha',\beta'}$$

which will automatically be an isomorphism because both sides are H_N-cleft extensions of k. Now an algebra homomorphism

$$\delta_0\colon A_{N,\alpha\alpha',\beta\alpha'+\beta'} \to A_{N,\alpha,\beta} \otimes A_{N,\alpha',\beta'}$$

can be defined by $\delta_0(x) = x \otimes x$ and $\delta_0(y) = 1 \otimes y + y \otimes x$, since

$$(\delta_0(x))^N = (x \otimes x)^N = x^N \otimes x^N = \alpha\alpha' 1 \otimes 1,$$

$$(\delta_0(y))^N = (1 \otimes y + y \otimes x)^N = 1 \otimes y^N + y^N \otimes x^N = (\beta' + \beta\alpha')1 \otimes 1$$

and

$$\delta_0(y)\delta_0(x) = (1 \otimes y + y \otimes x)(x \otimes x) = q(x \otimes x)(1 \otimes y + y \otimes x) = q\delta_0(x)\delta_0(y).$$

It is easy to check that δ_0 has its image in the cotensor product over H_N and gives rise to a left and right colinear map δ as desired. ∎

3. Hopf-Galois objects over tensor products

In this section we will be concerned with describing all Galois objects over a tensor product of two Hopf algebras. We will prove a generalization of part of a theorem of Kreimer, [6, Thm. 3.7], which says that the Harrison group of a tensor product of two cocommutative Hopf algebras is

$$\mathrm{Gal}(H_1 \otimes H_2) \cong \mathrm{Gal}(H_1) \oplus \mathrm{Gal}(H_2) \oplus \mathrm{Hopf}(H_2, H_1^*).$$

Note that the group of Hopf algebra homomorphisms $\mathrm{Hopf}(H_2, H_1^*)$ can be identified with the group $\mathrm{Pair}(H_2, H_1)$ of Hopf pairings $H_2 \otimes H_1 \to k$. In this notation, Kreimer's isomorphism maps (A_1, A_2, τ) to $A_1 \otimes A_2$ endowed with the obvious right $H_1 \otimes H_2$-comodule structure and the multiplication defined by

$$(x \otimes y)(x' \otimes y') := x\tau(y_{(1)}, {x'}_{(1)}){x'}_{(0)} \otimes y_{(0)}y'.$$

We will generalize this description of Galois objects over tensor product Hopf algebras to the non-cocommutative case (in which, however, there is no group structure on $\mathrm{Gal}(H_1 \otimes H_2)$).

To keep our results more general, we will, for this section only, assume that k is any commutative ring; all Hopf algebras considered will be flat k-modules, and Hopf-Galois objects will be faithfully k-flat by definition.

Recall that, for an H-comodule algebra A, a Hopf module $M \in \mathcal{M}_A^H$ is a right A-module as well as right H-comodule whose module structure is an H-colinear map: $\rho(ma) = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}$ for $a \in A$ and $m \in M$. If A is an H-Galois object, then Schneider's structure theorem for Hopf modules [16, Thm. 3.7] gives a category equivalence $\mathcal{M}_A^H \cong {}_k\mathcal{M}$; it takes a Hopf module to its submodule of coinvariants, and it takes a k-module V to $V \otimes A$ with the obvious Hopf module structure. In particular, the module structure of $M \in \mathcal{M}_A^H$ induces an isomorphism $M^{\mathrm{co}\,H} \otimes A \to M$.

Our first observation generalizes [6, Thm. 2.5 and Prop. 2.6.], where finite projective Hopf algebras are considered. Let $H = H_1 \otimes H_2$ be a tensor product of two Hopf algebras that are faithfully flat over k. Note that H_i can be considered as a Hopf subalgebra as well as quotient Hopf algebra of H. It is straightforward to check that for any right H-comodule V one has

$$V^{\mathrm{co}\,H_1} = V(H_2) := \{v \in V \mid v_{(0)} \otimes v_{(1)} \in V \otimes H_2\} \cong V \square_H H_2,$$

and vice versa.

LEMMA 3.1: *Let H_1, H_2 be two k-flat Hopf algebras and $H := H_1 \otimes H_2$. Let A be an H-Galois object.*

Then $A_i := A(H_i)$ is an H_i-Galois object for $i = 1, 2$, and multiplication induces an isomorphism

$$A(H_1) \otimes A(H_2) \cong A$$

of right H-comodules.

Proof: In fact A_i is an H_i-Galois object and a faithfully flat k-module by [15, Rem. 3.11]. Now consider A as a Hopf module in $\mathcal{M}_{A_2}^{H_2}$; then by the structure theorem of Hopf modules [15, Thm. 3.7] multiplication induces an isomorphism $A^{\mathrm{co}\,H_2} \otimes A_2 \to A$. ∎

Our next task is to describe, in the situation of Lemma 3.1, the algebra structure on $A_1 \otimes A_2$ resulting from that of A.

We first review the definition of skew pairings between Hopf algebras and the way they give rise to Hopf algebra cocycles. This was studied in [4] with applications to the Drinfeld double and other quantum group constructions (however, we switch the order of tensorands in a skew pairing).

Definition 3.2: Let L, H be two bialgebras. A skew pairing of L and H is a map $\tau: L \otimes H \to k$ satisfying

$$\tau(\ell\ell', h) = \tau(\ell, h_{(2)})\tau(\ell', h_{(1)}),$$
$$\tau(\ell, hh') = \tau(\ell_{(1)}, h)\tau(\ell_{(2)}, h'),$$

and $\tau(\ell, 1) = \varepsilon(\ell)$, $\tau(1, h) = \varepsilon(h)$ for all $\ell, \ell' \in L$ and $h, h' \in H$. A skew pairing is said to be invertible if it is as an element of $(L \otimes H)^*$. The trivial skew pairing is by definition $\varepsilon := \varepsilon_{L \otimes H}$.

Note that if H is a Hopf algebra then any skew pairing $\tau: L \otimes H \to k$ is invertible with $\tau^{-1} = \tau(\mathrm{id}_L \otimes S)$.

Remark 3.3: Let $\tau: L \otimes H \to k$ be an (invertible) skew pairing. Then $\hat{\tau} := \varepsilon \otimes \tau \otimes \varepsilon: (H \otimes L) \otimes (H \otimes L) \to k$ is an (invertible) two-cocycle.

As in [4] we denote the twisted Hopf algebra $(H \otimes L)^{\hat{\tau}}$ by $H \bowtie_\tau L$. Its multiplication is given by $(h \bowtie 1)(1 \bowtie \ell) = h \bowtie \ell$ and

$$(1 \bowtie \ell)(h \bowtie 1) = \tau(\ell_{(1)}, h_{(1)})(h_{(2)} \bowtie \ell_{(2)})\tau^{-1}(\ell_{(3)}, h_{(3)}).$$

LEMMA 3.4: *Let A be an L-H-bicomodule algebra and $\sigma\colon L \otimes L \to k$ a two-cocycle. Then $k_\sigma[A]$, defined to be A with the multiplication*

$$x \cdot y := \sigma(x_{(-1)}, y_{(-1)})x_{(0)}y_{(0)}$$

is a right H-comodule algebra. If σ is invertible, then $k_\sigma[A]$ is an L^σ-H-bicomodule algebra.

The proof consists of straightforward computations.

Definition 3.5: Let L_i be a Hopf algebra and A_i a left L_i-comodule algebra for $i = 1, 2$. Let $\tau\colon L_2 \otimes L_1 \to k$ be a skew pairing. Then we define $A_1 \#_\tau A_2 := k_{\hat{\tau}}[A_1 \otimes A_2]$. If H_i are also Hopf algebras such that A_i are L_i-H_i-bicomodule algebras, then Remark 3.3 and Lemma 3.4 imply that $A_1 \#_\tau A_2$ is an $L_1 \bowtie_\tau L_2$-$H_1 \otimes H_2$-bicomodule algebra.

For the following lemma, we will need some facts on the unique Hopf algebra $L(A, H)$ for which a given H-Galois object A is an $L(A, H)$-H-biGalois object. It can be constructed as $L(A, H) := L := (A \otimes A)^{\mathrm{co}\,H}$, which is a subalgebra of $A \otimes A^{\mathrm{op}}$. We will use the notation $\xi = \xi^{(1)} \otimes \xi^{(2)} \in A \otimes A$ for an element $\xi \in L$. Then the comultiplication of L is given by $\Delta(\xi) = (\xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]}) \otimes (\xi^{(1)}{}_{(1)}{}^{[2]} \otimes \xi^{(2)})$, where for $h \in H$ we denote the image of $1 \otimes h$ under the inverse of the Galois map $\kappa\colon A \otimes A \to A \otimes H$ by $h^{[1]} \otimes h^{[2]} \in A \otimes A$. Note that $h^{[1]}h^{[2]} = \varepsilon(h)$. By [12, Lem. 3.2, Lem. 3.3] L can be characterized by a universal property: Any H-colinear map $\delta\colon A \to W \otimes A$ factors as $(A \otimes f)\lambda$ for a unique k-module map $f\colon L \to W$. If W is a coalgebra (algebra, bialgebra), then δ is a comodule structure (algebra map, comodule algebra structure) if and only if f is a coalgebra map (algebra map, bialgebra map). In case V in the following lemma is finite projective, the lemma is a consequence of these results of [12], otherwise it is a generalization.

LEMMA 3.6: *Let H be a k-Hopf algebra and A an H-Galois object. Put $L := L(A, H)$. Then for all k-modules V, W there is a bijection*

$$\mathrm{Hom}^H(V \otimes A, W \otimes A) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Hom}_k(V \otimes L, W)$$

given by $\Phi(\alpha)(v \otimes \xi) \otimes 1_A = \alpha(v \otimes \xi^{(1)})\xi^{(2)}$ and $\Psi(\tau)(v \otimes a) = \tau(v \otimes a_{(-1)}) \otimes a_{(0)}$.

Assume that $W = k$ and let α and τ satisfy $\alpha = \Psi(\tau)$. If V is an algebra, then α is a module structure if and only if τ fulfills

$$\tau(vv' \otimes \xi) = \tau(v \otimes \xi_{(2)})\tau(v' \otimes \xi_{(1)})$$

for all $v, v' \in V$ and $\xi \in L$. If V is a coalgebra, then α is a measuring if and only if τ satisfies

$$\tau(v \otimes \xi\zeta) = \tau(v_{(1)} \otimes \xi)\tau(v_{(2)} \otimes \zeta)$$

for all $v \in V$ and $\xi, \zeta \in L$. In particular, if V is a bialgebra, then α gives A the structure of a V-module algebra if and only if τ is a skew pairing.

Proof: From Schneider's structure theorem [15, Thm. 3.7] for Hopf modules we have the following bijection:

$$\begin{aligned}\mathrm{Hom}^H(V \otimes A^{\cdot}, W \otimes A^{\cdot}) &\cong \mathrm{Hom}^H_{-A}(V \otimes A^{\cdot} \otimes A_{\cdot}^{\cdot}, W \otimes A_{\cdot}^{\cdot}) \\ &\cong \mathrm{Hom}_k((V \otimes A^{\cdot} \otimes A_{\cdot}^{\cdot})^{\mathrm{co}\, H}, W) = \mathrm{Hom}_k(V \otimes L, W)\end{aligned}$$

which is easily checked to have the claimed form. (In the calculation, we have indicated module and comodule structures on tensor products by dots; in particular, the comodule structure on $A^{\cdot} \otimes A^{\cdot}$ means the codiagonal comodule structure $\rho(x \otimes y) = x_{(0)} \otimes y_{(0)} \otimes x_{(1)}y_{(1)}$.)

From now on let $W = k$.

Assume that V is an algebra. If α is a module structure, then

$$\begin{aligned}\tau(vv' \otimes \xi)1_A &= \alpha(vv' \otimes \xi^{(1)})\xi^{(2)} \\ &= \alpha(v \otimes \alpha(v' \otimes \xi^{(1)}))\xi^{(2)} \\ &= \tau(v \otimes \alpha(v' \otimes \xi^{(1)}) \otimes \xi^{(2)})1_A \\ &= \tau(v \otimes \alpha(v' \otimes \xi^{(1)}{}_{(0)})\xi^{(1)}{}_{(1)}{}^{[1]}\xi^{(1)}{}_{(1)}{}^{[2]} \otimes \xi^{(2)})1_A \\ &= \tau(v \otimes \tau(v' \otimes \xi^{(1)}{}_{(0)} \otimes \xi^{(1)}{}_{(1)}{}^{[1]})\xi^{(1)}{}_{(1)}{}^{[2]} \otimes \xi^{(2)})1_A \\ &= \tau(v \otimes \xi_{(2)})\tau(v' \otimes \xi_{(1)})1_A.\end{aligned}$$

Conversely, if τ fulfills this equation, then we have

$$\begin{aligned}\alpha(v \otimes \alpha(v' \otimes a)) &= \alpha(v \otimes \tau(v' \otimes a_{(-1)})a_{(0)}) \\ &= \tau(v \otimes a_{(-1)})\tau(v' \otimes a_{(-2)})a_{(0)}.\end{aligned}$$

Now assume that V is a coalgebra. If α measures, then

$$\begin{aligned}\tau(v \otimes \xi\zeta)1_A &= \tau(v \otimes \xi^{(1)}\zeta^{(1)} \otimes \zeta^{(2)}\xi^{(2)}) \otimes 1 \\ &= \alpha(v \otimes \xi^{(1)}\zeta^{(1)})\zeta^{(2)}\xi^{(2)} \\ &= \alpha(v_{(1)} \otimes \xi^{(1)})\alpha(v_{(2)} \otimes \zeta^{(1)})\zeta^{(2)}\xi^{(2)} \\ &= \alpha(v_{(1)} \otimes \xi^{(1)})\xi^{(2)}\tau(v_{(2)} \otimes \zeta) \\ &= \tau(v_{(1)} \otimes \xi)\tau(v_{(2)} \otimes \zeta)1_A\end{aligned}$$

and conversely, if τ satisfies this equation, then

$$\begin{aligned}\alpha(v \otimes xy) &= \tau(v \otimes x_{(-1)}y_{(-1)})x_{(0)}y_{(0)} \\ &= \tau(v_{(1)} \otimes x_{(-1)})\tau(v_{(2)} \otimes y_{(-1)})x_{(0)}y_{(0)} \\ &= \alpha(v_{(1)} \otimes x)\alpha(v \otimes {}_{(2)} \otimes y). \qquad \blacksquare\end{aligned}$$

PROPOSITION 3.7: *Let H_1 and H_2 be two k-flat Hopf algebras.*

1. *Let A_i for $i = 1, 2$ be H_i-Galois objects, put $L_i := L(A_i, H_i)$ and let $\tau\colon L_2 \otimes L_1 \to k$ be a skew pairing. Then $A_1 \#_\tau A_2$ is an $H_1 \otimes H_2$-Galois object with $L(A_1 \# A_2, H_1 \otimes H_2) = L_1 \bowtie_\tau L_2$.*
2. *Let A be a right $H_1 \otimes H_2$-Galois object. Then there are unique up to isomorphism H_i-Galois objects A_i for $i = 1, 2$ and a skew pairing $\tau\colon L(A_2, H_2) \otimes L(A_1, H_1) \to k$, unique up to composition with coinner automorphisms of $L(A_2, H_2)$ and $L(A_1, H_1)$, such that $A \cong A_1 \#_\tau A_2$.*

Proof: To see that $A_1 \#_\tau A_2$ is a Galois object, one checks that the diagram

$$\begin{array}{ccc} A_1\#_\tau A_2 \otimes A_1\#_\tau A_2 & \xrightarrow{\kappa_{A_1\#A_2}} & A_1\#_\tau A_2 \otimes H_1 \otimes H_2 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle(2\,3)} \\ A_1 \otimes A_1 \otimes A_2 \otimes A_2 & \xrightarrow[\kappa_{A_1} \otimes \kappa_{A_2}]{} & A_1 \otimes H_1 \otimes A_2 \otimes H_2 \end{array}$$

commutes, where the right hand vertical arrow just switches tensorands, and α, defined by

$$\alpha(x\#y \otimes x'\#y') = x\tau(y_{(-1)}, x'_{(-1)}) \otimes x'_{(0)} \otimes y_{(0)} \otimes y',$$

is an isomorphism because τ is invertible.

That $A_1 \#_\tau A_2$ is a left $L_1 \bowtie_\tau L_2$-Galois object follows from the commutative diagram

$$\begin{array}{ccc} A_1\#_\tau A_2 \otimes A_1\#_\tau A_2 & \xrightarrow{\kappa'_{A_1\#A_2}} & L_1 \bowtie_\tau L_2 \otimes A_1\#_\tau A_2 \\ {\scriptstyle(2\,3)}\downarrow & & \downarrow{\scriptstyle\gamma} \\ A_1 \otimes A_1 \otimes A_2 \otimes A_2 & \xrightarrow[\kappa'_{A_1} \otimes \kappa'_{A_2}]{} & L_1 \otimes A_1 \otimes L_2 \otimes A_2 \end{array}$$

in which $\kappa'_R\colon R \otimes R \to B \otimes R$ denotes the left version of the Galois map for a left B-comodule algebra R, and γ, defined by

$$\gamma(\xi \otimes \zeta \otimes x \otimes y) = \xi_{(1)} \otimes x \otimes \zeta_{(1)}\tau(\zeta_{(2)}, \xi_{(2)}) \otimes y,$$

is a bijection because τ is invertible.

Note that the A_i in 1. are uniquely determined up to isomorphism, since clearly $A_i \cong (A_1\#_\tau A_2)(H_i)$.

Now let A be an H-Galois object for $H = H_1 \otimes H_2$. From Lemma 3.1 we have the isomorphism, induced by multiplication, $A_1 \otimes A_2 \to A$, with $A_i = A(H_i)$. Put $L_i := L(A_i, H_i)$.

We consider the category ${}_{A_2}\mathcal{M}^{H_2}_{A_2}$ of Hopf bimodules. By definition it consists of right H_2-comodules and A_2-bimodules that satisfy the compatibility conditions of a Hopf module in $\mathcal{M}^{H_2}_{A_2}$ as well as ${}_{A_2}\mathcal{M}^{H_2}$. The category ${}_{A_2}\mathcal{M}^{H_2}_{A_2}$ is a monoidal category with the tensor product over A_2, endowed with the obvious A_2-bimodule structure and the codiagonal H_2-comodule structure. In [13] we have proved that $({}_{A_2}\mathcal{M}^{H_2}_{A_2}, \otimes_{A_2}) \cong ({}_{L_2}\mathcal{M}, \otimes)$ as monoidal categories. The equivalence maps a left L_2-module V to the right Hopf module $V \otimes A_2$ with the left A_2-module structure $x(v\otimes y) = x_{(-1)} \cdot v \otimes x_{(0)}$. The inverse isomorphism maps $M \in {}_{A_2}\mathcal{M}^{H_2}_{A_2}$ to $M^{\mathrm{co}\,H_2}$ with the $L(A_2, H_2)$-module structure $\ell \cdot m = \ell^{(1)} m \ell^{(2)}$. Consider A as an H_2-comodule algebra. Then A_2 is a subcomodule algebra, which means that A is an algebra in the monoidal category ${}_{A_2}\mathcal{M}^{H_2}_{A_2}$ It follows that there is a unique left L_2-module structure on A_1 making it an L_2-module algebra, for which the multiplication on A satisfies $(xy)(x'y') = x(y_{(-1)} \cdot x')y_{(0)}y'$ for $x, x' \in A_1$ and $y, y' \in A_2$. This left L_2-action is given by $\ell \cdot x = \ell^{(1)} x \ell^{(2)}$, hence is H_1-colinear as a map $L_2 \otimes A_1 \to A_1$. Lemma 3.6 applies to yield a unique skew pairing τ: $L_2 \otimes L_1 \to k$ with $\ell \cdot x = \tau(\ell, x_{(-1)})x_{(0)}$, hence $xyx'y' = x\tau(y_{(-1)}, {x'}_{(-1)}){x'}_{(-1)}x_{(0)}y_{(0)}y'$ for all $x, x' \in A_1$ and $y, y' \in A_2$.

Now assume that we have another skew pairing χ: $L_2 \otimes L_1 \to k$ and an isomorphism f: $A \to A_1\#_\chi A_2$. By restriction, f induces automorphisms f_i of A_i with $f(xy) = f_1(x)\#f_2(y)$ for $x \in A_1$ and $y \in A_2$. By the universal property of L_i, there are algebra maps u_i: $L_i \to k$ with $f_i(a) = u_i(a_{(-1)})a_{(0)}$ for all $a \in A_i$. It follows that

$$\begin{aligned} yx' =& f^{-1}(f(y)f(x')) = f^{-1}(u_2(y_{(-1)})u_1({x'}_{(-1)})(1\#y_{(0)})({x'}_{(0)}\#1)) \\ =& \ u_2(y_{(-2)})u_1({x'}_{(-2)})\chi(y_{(-1)}, {x'}_{(-1)})f^{-1}({x'}_{(0)}\#y_{(0)}) \\ =& u_2(y_{(-3)})u_1({x'}_{(-3)})\chi(y_{(-2)}, {x'}_{(-2)})u_1^{-1}({x'}_{(-1)})u_2^{-1}(y_{(-1)}){x'}_{(0)}y_{(0)} \\ =& \ (\chi \circ (\mathrm{coinn}(u_2^{-1}) \otimes \mathrm{coinn}(u_1^{-1})))(y_{(-1)} \otimes {x'}_{(-1)}){x'}_{(0)}y_{(0)} \end{aligned}$$

for $y \in A_2$ and $x' \in A_1$, so that $\chi \circ (\mathrm{coinn}(u_2^{-1}) \otimes \mathrm{coinn}(u_1^{-1})) = \tau$ by uniqueness.

Conversely, if $\chi := \tau \circ (\mathrm{coinn}(u_2) \otimes \mathrm{coinn}(u_1))$, then essentially the same calculation shows that f: $A \to A_1\#_\chi A_2$, defined by

$$f(xy) = u_1(x_{(-1)})u_2(y_{(-1)})x_{(0)}\#y_{(0)},$$

is an isomorphism of H-comodule algebras. ∎

In general it is hard to say what the group $\mathrm{BiGal}(L_1 \otimes L_2, H_1 \otimes H_2)$ of bi-Galois objects between two tensor product Hopf algebras is. However, we can note the easy observation:

Remark 3.8: Let H_i, L_i for $i = 1, 2$ be Hopf algebras. Then we have an injective map

$$\mathrm{BiGal}(L_1, H_1) \times \mathrm{BiGal}(L_2, H_2) \to \mathrm{BiGal}(L_1 \otimes L_2, H_1 \otimes H_2)$$
$$(A_1, A_2) \mapsto A_1 \otimes A_2$$

which is a group homomorphism if $L_i = H_i$ for $i = 1, 2$.

4. Galois and bi-Galois objects over the general Taft algebras

We return to the case that k is a field containing a primitive N-th root of unity q. We will determine all Galois and bi-Galois objects over the Taft algebras with more than one grouplike generator. These are defined [16] to be

$$H_{N,m} := k\langle X_0, \ldots, X_{m-1}, Y\rangle / (X_i^N - 1, Y^N, YX_i - qX_iY)$$

with grouplike elements X_i and $(1, X_0)$-primitive Y.

We let C_N denote the cyclic group of order N. The group algebra kC_N^m has commuting grouplike generators $X_1, \ldots, X_m$ with defining relations $X_i^N = 1$. It is a selfdual Hopf algebra, with the isomorphism D: $kC_N^m \cong (kC_N^m)^*$ determined by $D(X_i)(X_j) = q^{\delta_{ij}}$).

We can use the results from the preceding section to compute Galois objects over $H_{N,m}$ in view of the following simple observation (that may well be known):

LEMMA 4.1: *There is an isomorphism of Hopf algebras*

$$f\colon H_{N,m+1} \cong kC_N^m \otimes H_N$$

determined by $f(X_0) = 1 \otimes X$, $f(X_i) = X_i \otimes X$ *for* $i \geq 1$ *and* $f(Y) = 1 \otimes Y$.

Hence, instead of the Taft algebras $H_{N,m}$ we can consider the tensor product Hopf algebra $kC_N^m \otimes H_N$. We will write $X_0 := X := 1 \otimes X$ and $X_i := X_i \otimes 1$.

Let us first fix some notations. In the following, in all sums and products the index runs through $1, \ldots, m$. In a noncommutative ring, products will be taken in ascending order of the indices from left to right.

We will abbreviate $\mathrm{GL}_m := \mathrm{GL}_m(\mathbb{Z}/(N))$. This group acts naturally by right matrix multiplication on G^m for any $\mathbb{Z}/(N)$-module G. If G is a multiplicative abelian group of exponent a divisor of N, then this right action reads $\boldsymbol{\alpha} \leftharpoonup T = (\prod_j \alpha_j^{t_{ji}})_i$ for $\boldsymbol{\alpha} = (\alpha_i) \in G^m$ and $T = (t_{ij}) \in \mathrm{GL}_m$.

We denote by $\mathrm{Skew}_m := \mathrm{Skew}_m(\mathbb{Z}/(N))$ the set of all skew symmetric matrices $R = (r_{ij}) \in M_m(\mathbb{Z}/(N))$, i.e. of matrices that satisfy $r_{ii} = 0$ and $r_{ij} = -r_{ji}$ for all i, j. The abelian group Skew_m has a right GL_m-action defined by $R \leftharpoonup T = (\sum_{k,\ell} r_{k\ell} t_{ki} t_{\ell j})_{i,j}$. (The group Skew_m is naturally isomorphic to the dual group $\left(\wedge^2(\mathbb{Z}/(N))^m\right)^*$ of the exterior square of $(\mathbb{Z}/(N))^m$. With this identification, the right action of GL_m is the dual of the canonical left action on $\wedge^2(\mathbb{Z}/(N))^m$.)

We will now describe the groups of Galois and bi-Galois objects over kC_N^m.

LEMMA 4.2: *The map*

$$\mathrm{Skew}_m \oplus (\dot{k}/\dot{k}^N)^m \ni (R, \boldsymbol{\alpha}) \longmapsto B_{R,\boldsymbol{\alpha}} \in \mathrm{Gal}(kC_N^m)$$

defined by

$$B_{R,\boldsymbol{\alpha}} := k\langle x_1, \ldots, x_m \rangle / (x_i^N - \alpha_i, x_j x_i - q^{r_{ij}} x_j x_i)$$

with the right kC_N^m-comodule algebra structure determined by $\rho(x_i) = x_i \otimes X_i$ is an isomorphism of abelian groups.

If N is odd, then the map

$$\mathrm{GL}_m(\mathbb{Z}/(N)) \ltimes \left(\mathrm{Skew}_m \oplus (\dot{k}/\dot{k}^N)^m\right) \to \mathrm{BiGal}(kC_N^m)$$
$$(T, R, \boldsymbol{\alpha}) \mapsto B_{T,R,\boldsymbol{\alpha}}$$

defined by $B_{T,R,\boldsymbol{\alpha}} = B_{R,\boldsymbol{\alpha}}$ as right comodule algebras with left comodule structure defined by

$$\lambda(x_i) = \prod_j X_j^{t_{ji}} \otimes x_i$$

is a group isomorphism.

Proof: The description of $\mathrm{Gal}(kC_N^m)$ is probably well known. We will sketch how to deduce it from Kreimer's result cited at the beginning of the preceding section. First, consider a kC_N-Galois object A. This is a strongly graded ring, and has a normal basis, so it is easy to see that it has the form $A \cong k[x]/(x^N - \alpha)$ for some $\alpha \in \dot{k}$. Any isomorphism $k[x]/(x^N - \alpha) \to k[x]/(x^N - \alpha')$ maps x to λx for some $\lambda \in \dot{k}$, and one checks that $x \mapsto \lambda x$ extends to a well defined isomorphism iff $\alpha = \lambda^N \alpha'$. This is the desired description of $\mathrm{Gal}(kC_N^1)$. By induction over [6, Thm. 3.7], using $\mathrm{Pair}(kC_N^\ell, kC_N) \cong \mathrm{Pair}(kC_N, kC_N)^\ell \cong (\mathbb{Z}/(N))^\ell$, one obtains an isomorphism

$$\mathrm{Gal}(kC_N^m) \cong \mathrm{Gal}(kC_N)^m \oplus (\mathbb{Z}/(N))^{m(m-1)/2}.$$

Identifying $(\mathbb{Z}/(N))^{m(m-1)/2}$ with Skew_m yields the claim on $\mathrm{Gal}(kC_N^m)$.

In [12, Lem. 4.7] we have shown that there is an isomorphism

$$\mathrm{Aut}_{\mathrm{Hopf}}(H) \ltimes \mathrm{Gal}(H) \ni (f, A) \mapsto {}^f A \in \mathrm{BiGal}(H)$$

for any cocommutative Hopf algebra H; the action of $\mathrm{Aut}_{\mathrm{Hopf}}(H)$ on $\mathrm{Gal}(H)$ is given by $A \leftharpoonup f = A^{f^{-1}}$. In our case, we clearly have $\mathrm{Aut}_{\mathrm{Hopf}}(kC_N^m) \cong \mathrm{GL}_m(\mathbb{Z}/(N))$. We choose the identification

$$\mathrm{GL}_m(\mathbb{Z}/(N)) \ni T \mapsto F_T \in \mathrm{Aut}_{\mathrm{Hopf}}(kC_N^m)$$

given by $F_T(X_i) = \prod_j X_j^{t_{ji}}$. Assume N is odd. We have to show $B_{R,\boldsymbol{\alpha}}^{F_T^{-1}} \cong B_{R\leftharpoonup T,\boldsymbol{\alpha}\leftharpoonup T}$ and ${}^{F_T}B_{R,\boldsymbol{\alpha}} \cong B_{T,R,\boldsymbol{\alpha}}$. The latter is straightforward, the former is proved by considering the isomorphism

$$G_T\colon B_{R\leftharpoonup T,\boldsymbol{\alpha}\leftharpoonup T} \to B_{R,\boldsymbol{\alpha}}^{F_T^{-1}}$$

determined by $G_T(x_i) = \prod_j x_j^{t_{ji}}$.

To see that this is really well defined, one needs the following fact: Let a, b be elements in a k-algebra R satisfying $ba = uab$ for some $u \in k$. Then one proves by induction that $(ab)^N = u^{\frac{1}{2}N(N-1)}a^N b^N$, so that, if N is odd and u is any N-th root of unity, then $(ab)^N = a^N b^N$. ∎

LEMMA 4.3: *For every* $\mathbf{r} \in (\mathbb{Z}/(N))^m$ *there is a unique skew pairing* $\tau_{\mathbf{r}}\colon H_N \otimes kC_N^m \to k$ *with* $\tau_{\mathbf{r}}(X, X_i) = q^{-r_i}$ *and* $\tau_{\mathbf{r}}(Y, X_i) = 0$ *for all* i. *Every skew pairing* $\tau\colon H_N \otimes kC_N^m \to k$ *equals* $\tau_{\mathbf{r}}$ *for a unique* $\mathbf{r} \in (\mathbb{Z}/(N))^m$.

The proof is immediate, since a skew pairing τ is the same as a Hopf algebra homomorphism $H_N \to kC_N^m$ in view of the self-duality of kC_N^m, and there is no noncentral skew primitive element in kC_N^m.

For the following application of Proposition 3.7, note that the coinner automorphisms of H_N fix X, hence do not affect the skew pairing $\tau_{\mathbf{r}}$, while kC_N^m has only trivial coinner automorphisms.

COROLLARY 4.4: *There is a bijection*

$$\begin{aligned}(\mathbb{Z}/(N))^m \times \mathrm{Skew}_m \times (\dot{k}/\dot{k}^N) \times k \times (\dot{k}/\dot{k}^N)^m &\to \mathrm{Gal}(kC_N^m \otimes H_N)\\ (R, \mathbf{r}, \alpha_0, \beta, \boldsymbol{\alpha}) &\mapsto \Gamma_{R,\mathbf{r},\alpha_0,\beta,\boldsymbol{\alpha}}\end{aligned}$$

defined by

$$\begin{aligned}\Gamma_{R,\mathbf{r},\alpha_0,\beta,\boldsymbol{\alpha}} :=& B_{R,\boldsymbol{\alpha}} \#_{\tau_{\mathbf{r}}} A_{\alpha_0,\beta}\\ \cong& k\langle x_0, \dots, x_m, y\rangle / (x_i^N - \alpha_i, y^N - \beta, yx_i - q^{\delta_{i0}} x_i y, x_j x_i - q^{r_{ij}} x_i x_j)\end{aligned}$$

where we have put $r_0 := 1$ and $r_{0i} := r_i =: -r_{i0}$. The right comodule algebra structure is determined by $\rho(x_i) = x_i \otimes X_i$ and $\rho(y) = 1 \otimes Y + y \otimes X_0$.

Proof: One only has to check that $x_i := x_i\#1$ has the indicated commutation relations with $x_0 := 1\#x_0$ and $y := 1\#y$ in the twisted product. We have

$$(1\#x_0)(x_i\#1) = \tau_{\mathbf{r}}(X_0, X_i)x_i\#x_0 = q^{-r_i}(x_i\#1)(1\#x_0)$$

and

$$(1\#y)(x_i\#1) = \tau_{\mathbf{r}}(1, X_i)x_i\#y + \tau_{\mathbf{r}}(Y, X_i)x_i\#x_0 = x_i\#y = (x_i\#1)(1\#y). \quad ∎$$

We know by Proposition 3.7 that since $L(A, H_N) \cong H_N$ for any H_N-Galois object A and $L(B, kC_N^m) \cong kC_N^m$ for any kC_N^m-Galois object B by cocommutativity of kC_N^m, we have $L(\Lambda, kC_N^m \otimes H_N) \cong kC_N^m \bowtie_\tau H_N$ for a suitable skew pairing τ, for all $kC_N^m \otimes H_N$-Galois objects Λ.

LEMMA 4.5: *Let $\mathbf{r} \in (\mathbb{Z}/(N))^m$. There is a Hopf algebra isomorphism*

$$\begin{aligned} kC_N^m \bowtie_{\tau_{\mathbf{r}}} H_N &\cong kC_N^m \otimes H_N \\ X_i \bowtie 1 &\mapsto X_i \otimes X_0^{r_i} \\ 1 \bowtie X_0 &\mapsto 1 \otimes X_0 \\ 1 \bowtie Y &\mapsto 1 \otimes Y. \end{aligned}$$

Proof: In $kC_N^m \otimes H_N$ we have $(1\otimes Y)(X_i \otimes X_0^{r_i}) = X_i \otimes Y X_0^{r_i} = X_i \otimes q^{r_i} X_0^{r_i} Y = q^{r_i}(X_i \otimes X_0^{r_i})(1 \otimes Y)$. In $kC_N^m \bowtie_{\tau_{\mathbf{r}}} H_N$ we have

$$\begin{aligned}(1 \bowtie Y)(X_i \bowtie 1) &= \tau_{\mathbf{r}}(1, X_i)X_i \bowtie 1\tau_{\mathbf{r}}^{-1}(Y, X_i) \\ &\quad + \tau_{\mathbf{r}}(1, X_i)X_i \bowtie Y\tau_{\mathbf{r}}^{-1}(X_0, X_i) \\ &\quad + \tau_{\mathbf{r}}(Y, X_i)X_i \bowtie X_0\tau_{\mathbf{r}}^{-1}(X_0, X_i) \\ &= q^{r_i}(X_i \bowtie 1)(1 \bowtie Y).\end{aligned}$$

We omit the rest of the proof. ∎

COROLLARY 4.6: *Every right $kC_N^m \otimes H_N$-Galois object has the structure of a $kC_N^m \otimes H_N$-$kC_N^m \otimes H_N$-bi-Galois object. For every two-cocycle σ on $kC_N^m \otimes H_N$ we have $(kC_N^m \otimes H_N)^\sigma \cong kC_N^m \otimes H_N$. The isomorphism class of the Hopf algebra $kC_N^m \otimes H_N$ (the Taft algebra) is uniquely determined by the k-linear monoidal category of its comodules.*

In particular, we can choose the left comodule structure λ on $\Gamma_{R,\mathbf{r},\alpha_0,\beta,\boldsymbol{\alpha}}$ determined by $\lambda(x_0) = X_0 \otimes x_0, \lambda(y) = 1 \otimes y + Y \otimes x_0$ and

$$\lambda(x_i) = X_0^{r_i} X_i \otimes x_i$$

for $i > 0$ to make it a bi-Galois object.

After choosing a bi-Galois structure for each Galois object, all possible ones are obtained by twisting the left comodule structure with a set of representatives for the coouter automorphism group of kC_N^m. We will also want to determine the group structure of $\mathrm{BiGal}(kC_N^m \otimes H_N)$, at least for odd N. To do this, we first need some more facts and notations:

1. The group $(\mathbb{Z}/(N))^m$ acts on the group $(\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^m$ by
$$(\alpha_0, \beta, \boldsymbol{\alpha}) \leftharpoonup \mathbf{r} = (\alpha_0, \beta, (\alpha_i \alpha_0^{r_i})_i).$$
2. Letting GL_m act trivially on $\dot{k} \ltimes k$, we get an action of GL_m on $(\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)$.
3. The action of $(\mathbb{Z}/(N))^m$ on $(\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^m$ is GL_m-equivariant, that is
$$((\alpha_0, \beta, \boldsymbol{\alpha}) \leftharpoonup \mathbf{r}) \leftharpoonup T = ((\alpha_0, \beta, \boldsymbol{\alpha}) \leftharpoonup T) \leftharpoonup (\mathbf{r} \leftharpoonup T)$$
holds for all $T \in \mathrm{GL}_m$, $\mathbf{r} \in (\mathbb{Z}/(N))^m$. This ensures that we get a right action of $\mathrm{GL}_m \ltimes (\mathbb{Z}/(N))^m$ on $(\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^m$ defined by $\xi \leftharpoonup (T, \mathbf{r}) = (\xi \leftharpoonup T) \leftharpoonup \mathbf{r}$ for $\xi \in (\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^m$.
4. The map
$$\omega_0\colon (\mathbb{Z}/(N))^m \times (\mathbb{Z}/(N))^m \ni (\mathbf{r}, \mathbf{r}') \mapsto (r_i r_j' - r_j r_i')_{i,j} \in \mathrm{Skew}_m$$
is biadditive, hence a two-cocycle on the group $(\mathbb{Z}/(N))^m$ with values in the trivial module Skew_m. The map ω_0 is GL_m-equivariant, $\omega_0(\mathbf{r}, \mathbf{r}') \leftharpoonup T = \omega_0(\mathbf{r} \leftharpoonup T, \mathbf{r}' \leftharpoonup T)$, whence $\omega((T, \mathbf{r}), (T', \mathbf{r}')) := \omega_0(\mathbf{r} \leftharpoonup T', \mathbf{r}')$ defines a two-cocycle on $\mathrm{GL}_m \ltimes (Z/(N))^m$ with values in Skew_m.

THEOREM 4.7: *Assume N is odd. An isomorphism*
$$\Psi\colon (\mathrm{GL}_m \ltimes (\mathbb{Z}/(N))^m) \ltimes_\omega \left(\mathrm{Skew}_m \times (\dot{k} \ltimes k) \times (\dot{k}/\dot{k}^N)^m\right) \longrightarrow \mathrm{BiGal}(kC_N^m \otimes H_N)$$
is given by $\Psi((T, \mathbf{r}), (R, (\alpha_0, \beta), \boldsymbol{\alpha}) := \Lambda_{T,\mathbf{r},R,\alpha_0,\beta,\boldsymbol{\alpha}} := \Gamma_{\mathbf{r},R,\alpha_0,\beta,\boldsymbol{\alpha}}$ as a right $kC_N^m \otimes H_N$-comodule algebra, with the left comodule algebra structure determined by $\lambda(x_0) = X_0 \otimes x_0, \lambda(y) = 1 \otimes y + Y \otimes x_0$ and
$$\lambda(x_i) = X_0^{r_i} \prod_j X_j^{t_{ji}} \otimes x_i$$
for $i > 0$.

Proof: For a general Hopf algebra H, let $\mathcal{G}$ be a set of H-H-bi-Galois objects whose underlying right H-Galois objects are a representative system for $\mathrm{Gal}(H)$, and let $\mathcal{F}$ be a representative system for $\mathrm{CoOut}(H)$. Then by [12, Lem. 3.11.] we have a bijection

$$\mathcal{F} \times \mathcal{G} \ni (\theta, A) \mapsto {}^{\theta}A \in \mathrm{BiGal}(H).$$

For $\mathbf{r} \in (\mathbb{Z}/(N))^m$, $R \in \mathrm{Skew}_m$, $\beta \in k$, $\alpha_0 \in \dot{k}$ and $\boldsymbol{\alpha} \in (\dot{k}/\dot{k}^N)^m$ we define $\mathcal{J}(\mathbf{r}, R, \alpha_0, \beta, \boldsymbol{\alpha}) := \Gamma_{\mathbf{r},R,\alpha_0,\beta,\boldsymbol{\alpha}}$ with the left comodule algebra structure satisfying $\lambda(x_0) = X_0 \otimes x_0$, $\lambda(x_i) = X_0^{r_i} X_i \otimes x_i$ for $i > 0$ and $\lambda(y) = 1 \otimes y + Y \otimes x_0$.

Choose a representative system $\mathcal{A}$ for $\dot{k}/\dot{k}^N$. Then

$$\mathcal{J}\colon (\mathbb{Z}/(N))^m \times \mathrm{Skew}_m \times \mathcal{A} \times k \times (\dot{k}/\dot{k}^N)^m \to \mathrm{BiGal}(kC_N^m \otimes H_N)$$

is injective and yields a bijection when composed with the forgetful map

$$\mathrm{BiGal}(kC_N^m \otimes H_N) \to \mathrm{Gal}(kC_N^m \otimes H_N).$$

It is straightforward to check that the canonical map

$$\mathrm{Aut}_{\mathrm{Hopf}}(kC_N^m) \times \mathrm{CoOut}(H_N) \to \mathrm{CoOut}(kC_N^m \otimes H)$$

is a bijection. Hence, choosing a representative system $\mathcal{S}$ for $\dot{k}/\langle q\rangle$, the map

$$\mathcal{J}'\colon \mathrm{GL}_m \times \mathcal{S} \times (\mathbb{Z}/(N))^m \times \mathrm{Skew}_m \times \mathcal{A} \times k \times (\dot{k}/\dot{k}^N)^m \longrightarrow \mathrm{BiGal}(kC_N^m \otimes H_N)$$

given by $\mathcal{J}'(T, v, \mathbf{r}, R, \alpha_0, \beta, \boldsymbol{\alpha}) = {}^{F_T \otimes f_v}\mathcal{J}(\mathbf{r}, R, \alpha_0, \beta, \boldsymbol{\alpha})$ is a bijection. Similarly to the proof of Theorem 2.5 one shows that there is an isomorphism of $kC_N^m \otimes H_N$-bicomodule algebras

$$\begin{aligned} \mathcal{J}(\mathbf{r}, R, v^N\alpha_0, \beta, \boldsymbol{\alpha}) &\to {}^{\mathrm{id} \otimes f_v}\mathcal{J}(\mathbf{r}, R, \alpha_0, \beta, \boldsymbol{\alpha}) \\ x_0 &\mapsto v x_0 \\ x_i &\mapsto x_i \quad \text{for } i > 0 \\ y &\mapsto y. \end{aligned}$$

Since $\mathcal{S} \times \mathcal{A} \ni (v, \alpha) \mapsto v^N\alpha \in \dot{k}$ is a bijection, we obtain a bijection

$$\Psi\colon \mathrm{GL}_m \times (\mathbb{Z}/(N))^m \times \mathrm{Skew}_m \times \dot{k} \times k \times (\dot{k}/\dot{k}^N)^m \cong \mathrm{BiGal}(kC_N^m \otimes H_N)$$

by setting

$$\Psi(T, \mathbf{r}, R, \alpha_0, \beta, \boldsymbol{\alpha}) = {}^{F_T \otimes \mathrm{id}}\mathcal{J}(\mathbf{r}, R, \alpha_0, \beta, \boldsymbol{\alpha}),$$

which clearly has the form indicated in the formulation of the Theorem.

It remains to verify that Ψ is a group homomorphism. Let $T, T' \in \mathrm{GL}_m$, $\mathbf{r}, \mathbf{r}' \in (\mathbb{Z}/(N)^m$, $R, R' \in \mathrm{Skew}_m$, $\beta, \beta' \in k$, $\alpha_0, \alpha_0' \in \dot{k}$ and $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in (\dot{k}/\dot{k}^N)^m$. Set $T'' := TT'$, $\mathbf{r}'' := \mathbf{r} \leftharpoonup T' + \mathbf{r}'$,

$$r''_{ij} := \sum_{k,\ell} r_{k\ell} t'_{ki} t'_{\ell j} + \sum_k (r_k t'_{ki} r'_j - r_k t'_{kj} r'_i) + r'_{ij},$$

$\beta'' := \beta\alpha_0' + \beta'$, $\alpha_0'' := \alpha_0\alpha_0'$, and $\alpha_i'' := \alpha_0^{r_i'} \left(\prod_j \alpha_j^{t'_{ji}}\right) \alpha_i'$. We have to check

$$\Lambda_{T'',\mathbf{r}'',R'',\alpha_0'',\beta'',\boldsymbol{\alpha}''} \cong \Lambda_{T,\mathbf{r},R,\alpha_0,\beta,\boldsymbol{\alpha}} \Box_{kC_N^m \otimes H_N} \Lambda_{T',\mathbf{r}',R',\alpha_0',\beta',\boldsymbol{\alpha}'}.$$

We can define an isomorphism δ by $\delta(x_i) = x_0^{r_i'} \prod_j x_j^{t'_{ji}} \otimes x_i$ for $i > 0$, $\delta(x_0) = x_0 \otimes x_0$, and $\delta(y) = 1 \otimes y + y \otimes x_0$.

In fact $\delta(x_i)$ is in the cotensor product because

$$\begin{aligned}\rho(x_0^{r_i'} \prod_j x_j^{t'_{ji}}) \otimes x_i &= (x_0 \otimes X_0)^{r_i'} \prod_j (x_j \otimes X_j)^{t'_{ji}} \otimes x_i \\ &= x_0^{r_i'} \prod_j x_j^{t'_{ji}} \otimes X_0^{r_i'} \prod_j X_j^{t'_{ji}} \otimes x_i = x_0^{r_i'} \prod_j x_j^{t'_{ji}} \otimes \lambda(x_i).\end{aligned}$$

For $i > 0$ we have

$$\delta(x_i)^N = (x_0^{r_i'} \prod_j x_j^{t'_{ji}} \otimes x_i)^N = (x_0^N)^{r_i'} \prod_j (x_j^N)^{t'_{ji}} \otimes x_i^N = \alpha_0^{r_i'} \prod_j \alpha_j^{t'_{ji}} \alpha_i' (1 \otimes 1),$$

using again that for any elements $a, b \in R$ of a k-algebra R satisfying $ba = uab$ for some N-th root of unity in k, we have $(ab)^N = a^N b^N$. Moreover, for $i, j > 0$

$$\begin{aligned}\delta(x_j)\delta(x_i) &= \left(x_0^{r_j'} \prod_\ell x_\ell^{t'_{\ell j}} \otimes x_j\right)\left(x_0^{r_i'} \prod_k x_k^{t'_{ki}} \otimes x_i\right) \\ &= x_0^{r_j'} \left(\prod_\ell x_\ell^{t'_{\ell j}}\right) x_0^{r_i'} \left(\prod_k x_k^{t'_{ki}}\right) \otimes x_j x_i \\ &= q^{(r'_{ij} + \sum_\ell r_\ell t'_{\ell j} r'_i)}\, x_0^{r_j'} x_0^{r_i'} \left(\prod_\ell x_\ell^{t'_{\ell j}}\right)\left(\prod_k x_k^{t'_{ki}}\right) \otimes x_i x_j \\ &= q^{(r'_{ij} + \sum_\ell r_\ell t'_{\ell j} r'_i + \sum_{k,\ell} r_{k\ell} t'_{ki} t'_{\ell j})} \\ &\qquad \times x_0^{r_i'} x_0^{r_j'} \left(\prod_k x_k^{t'_{ki}}\right)\left(\prod_\ell x_\ell^{t'_{\ell j}}\right) \otimes x_i x_j \\ &= q^{(r'_{ij} + \sum_\ell r_\ell t'_{\ell j} r'_i + \sum_{k,\ell} r_{k\ell} t'_{ki} t'_{\ell j} - \sum r_k t_{kj} r'_j)} \\ &\qquad \times x_0^{r_i'} \left(\prod_k x_k^{t'_{ki}}\right) x_0^{r_j'} \left(\prod_\ell x_\ell^{t'_{\ell j}}\right) \otimes x_i x_j \\ &= q^{r''_{ij}}\, \delta(x_i)\delta(x_j).\end{aligned}$$

We omit the remaining details showing that δ is a well defined bicomodule algebra homomorphism. It is then an isomorphism since its domain and codomain are Galois objects. ∎

COROLLARY 4.8: *If N is odd and k is algebraically closed, then*

$$\mathrm{BiGal}(kC_N^m \otimes H_N) \cong ((\mathrm{GL}_m \ltimes (\mathbb{Z}/(N))^m) \ltimes_\omega \mathrm{Skew}_m) \times (\dot{k} \ltimes k).$$

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